Sharp lower bounds on the scalar curvature

Want to prove

Thm If (M^n, g, X, λ) is a complete Ricci soliton,

then 1) $R \geq 0$ if $\lambda \geq 0$

2) $R \geq \frac{\lambda n}{2}$ if $\lambda < 0$

moreover, if equality holds at any point of M then (M,g) is Einstein and if $\lambda > 0$ and $X = \nabla f$, $f \in C^\infty(M)$ then the soliton is the Gaussian shrinker.

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for (M^n, g) Riemannian, define for $\gamma : [a,b] \to M$

$$L(\gamma) = \int_a^b |\gamma'(r)| dr$$

$$d(x,y) = \inf_\gamma L(\gamma).$$

We want to find the 1^{st} variation of L. For $\gamma : [0, L] \to M$

$$\frac{d}{dv}\bigg|_{v=0} L(\gamma_v) = -\int_0^L \langle V(r), \nabla_{\gamma'} \gamma' \rangle dr + \langle V(r), \gamma'(r) \rangle \bigg|_{r=0}^{L}$$

where $V(r) = \frac{d}{dv}\Big|_{v=0} \gamma_v(r)$

Thm. The 2^{nd} variation of L is given by

$$\frac{d^2}{dv^2}\Big|_{v=0} L(\gamma_v) = \int_0^L \left[|(\nabla_{\gamma'(r)} V)^\perp|^2 - \langle R(V, \gamma'(r)), \gamma'(r), V\rangle \right] dr$$

$$+ \left\langle \nabla_V \frac{\partial}{\partial v} \gamma_v, \gamma'(L) \right\rangle$$

where $(\nabla_{\gamma'} V)^\perp = \nabla_{\gamma'} V - \langle \nabla_{\gamma'} V, \gamma' \rangle \gamma'$

and R is the Riemann curvature tensor.

<u>Def^n</u> $\varphi : M^n \to \mathbb{R}$ continuous in a nbd of $x \in M$. We say

i) $\Delta \varphi \leq A$ in <u>the barrier sense</u> if $\forall\, \varepsilon > 0$ $\exists$ C^2-function $\psi \geq \varphi$ w/ $\psi(x) = \varphi(x)$ and $\Delta \psi \leq A + \varepsilon$.

ii) " ——— " <u>in the strong barrier sense</u> if " ——— "

$\psi \geq \varphi$, $\psi(x) = \varphi(x)$ and $\Delta \psi \leq A$.

Fix $p \in M^n$, $r_x = r(x) = d(p, x)$. We want to bound the Laplacian of $d(x, p)$.

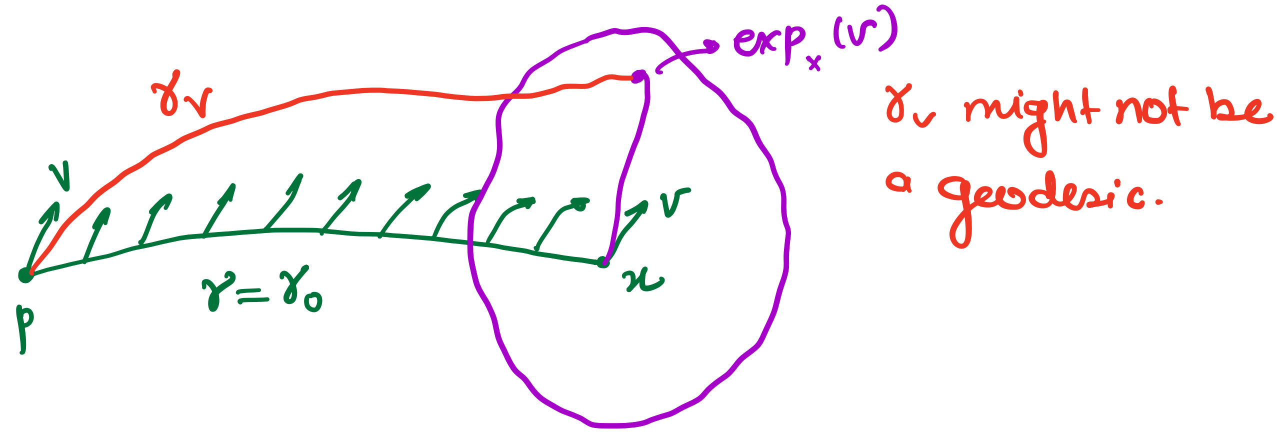
Prop Let $x \neq p$, $\gamma : [0, r_x] \longrightarrow M^n$ unit speed minimal geodesic joining p and x. Let $\xi : [0, r_x] \longrightarrow \mathbb{R}$ be a continuous, piecewise C^∞-function w/ $\xi(0) = 0$ and $\xi(r_x) = 1$.

Then

$$\Delta r(x) = \Delta d(x,p) \leq \int_0^{r_x} \left[(n-1)(\xi')^2(r) - \xi^2(r) Ric(\gamma', \gamma')\right] dr$$

in the strong barrier sense.

Proof:- $x \neq p$. Let ε be small enough, s.t. $\exp^g$ is injective in a nbd. of x, i.e., $\varepsilon \in (0, inj_g(x))$.



Define $v \in T_pM$, parallel transport along γ: and define

$$\gamma^v(r) = \exp_{\gamma(r)}(\xi(r) \cdot v(r)).$$ This has the following properties

- $\xi(0)=0 \Rightarrow \gamma^V(0)=p \quad \forall\, V.$
- $\gamma^0(r)=\gamma(r)$
- $\gamma^V(r_x)=\exp_x(V(r_x))$ as $\xi_{r_x}=1.$
- for a fixed r, $\left.\frac{\partial}{\partial t}\right|_{t=0}\gamma^{tV}(r)=\xi(r)\cdot V(r)$

As $\gamma=\gamma^0$ is a minimizing geodesic b/w p and x and $r_x=d(x,p) \Rightarrow L(\gamma^0)=r_x$.

Also

$$r_{\exp_x v} = d(p,\exp_x v) \leq L(\gamma^V)$$

$\because$ $\varepsilon\in(0,\mathrm{inj}_g x)$, $\exp_x^g : T_xM \supseteq B_\varepsilon(0) \longrightarrow B_\varepsilon(x)$ is a diffeo

$\Rightarrow$ for $y\in B_\varepsilon(x)$, $\exp_x^{-1}(y)$ exists and lies in $B_\varepsilon(0)$.

Define

$$\varphi(y)= L\left(\gamma^{\exp_x^{-1}y}\right).$$ Let's calculate $\Delta\varphi$.

Consider the o.n.b. $\{e_1,\dots e_{n-1},\gamma'(r_x)\}$ of T_xM. Parallel transport these vectors along γ to get a $\{e_1(r),\dots e_{n-1}(r),\gamma'(r)\}$ an o.n.b. for $T_{\gamma(r)}M$.

note, $\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma^{te_i}(r) = \xi(r)\cdot e_i(r)$.

Also, $(\nabla_{\gamma'} V)^\perp = \underbrace{\nabla_{\gamma'}(\xi(r).e_i(r))}_{\xi'(r).\,e_i(r)} - \overbrace{\langle \xi(r).e_i(r), \gamma'(r)\rangle}^{=0 \text{ b/c o.n.b.}}\, \gamma'(r)$

$\therefore$

$$\Delta\varphi(x) = \sum_{i=1}^{n-1} \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \varphi(\exp_x(te_i)) + \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \varphi(\exp_x(t.\gamma'(r_x)))$$

$$= \sum_{i=1}^{n-1} \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} L(\gamma^{te_i}) + 0 \quad \text{(b/c we are getting 2 derivatives w.r.t. } t\text{)}.$$

$$= \sum_{i=1}^{n-1} \int_0^{r_x} \left[(\xi'(r))^2 - \xi^2(r) \langle R(e_i, \gamma'(r))\gamma'(r), e_i\rangle \right] dr$$

$$= \int_0^{r_x} \left[(n-1)(\xi')^2(r) - \xi^2(r)\, Ric(\gamma'(r), \gamma'(r)) \right] dr. \quad \boxtimes$$

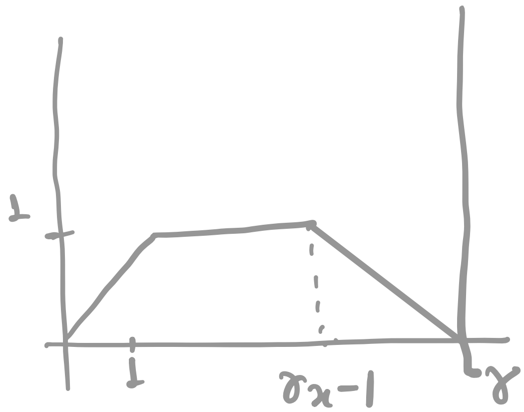
<u>Que.</u> What are good choices for ξ?

Let $\xi = \dfrac{r}{r_x}$.

Corr: For $Ric \geq 0$, $\Delta r(x) \leq \dfrac{n-1}{r(x)}$ in a barrier sense.

For $x \in M^n \setminus B_1(p)$ and $\gamma : [0, r_x] \to M$ be a unit speed geodesic, define

$$\xi(r) = \begin{cases} r, & 0 \leq r \leq 1 \\ 1, & 1 < r < r_x - 1 \\ r_x - r, & r_x - 1 < r \leq r_x \end{cases}$$



$\because$ γ is minimal and we choose the same o.n.b. as before to get

$$0 \leq \delta^2 L(\gamma^{\xi e_i}) = \int_0^{r_x} \left[(\xi')^2(r) - \xi^2(r) \langle R(e_i, \gamma'(r))\gamma'(r), e_i \rangle dr \right.$$

then summing over i, we get

$$\int_0^{r_x} \underbrace{\xi^2(r)} \mathrm{Ric}(\gamma'(r),\gamma'(r))\,dr \le (n-1)\int_0^{r_x} \underbrace{(\xi')^2(r)}_{\le 2}\,dr$$

want to get rid ξ^2 in the integral

Let $S(x) = \sup_{\substack{v \in S_y^{n-1} \\ y \in B_1(x)}} |\mathrm{Ric}(v,v)|$

Then we get

$$\int_0^{r_x} \mathrm{Ric}(\gamma'(r),\gamma'(r))\,dr \le 2(n-1) + \frac{2}{3}\left(S(p)+S(x)\right).$$

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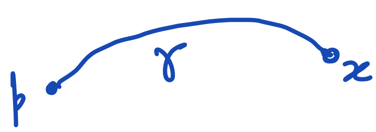
From last time

$$\int_0^{r_x} \mathrm{Ric}(\gamma'(r),\gamma'(r))\,dr \le 2(n-1) + \frac{2}{3}\left[S(p)+S(x)\right]$$

w/ γ a unit-speed minimizing geodesic.



and $S(y) = \sup_{\substack{v \in S_y^{n-1} \\ y \in B_1(x)}} \mathrm{Ric}(v,v)_+$

Def^n Define $\Delta_X \phi = \Delta\phi - \langle X, \nabla\phi\rangle$, $X \in \Gamma(TM)$, $\phi \in C^\infty(M)$.

Prop:- (M^n, g, X, λ) is a complete RS. Denote by $r(x) = d(p,x)$, $p \in M$. Assume $|Ric| \le K_0$ on $B_{r_0}(p)$. Then $\exists$ constant $C(n)$

s.t. $$\Delta_X r \le \underbrace{-\frac{\lambda}{2} r + C(n)\left(K_0 r_0 + r_0^{-1}\right) + |X|(p)}_{h}$$

in the barrier sense on $M^n \setminus B_{r_0}(p)$.

Proof:- Suppose $x \notin B_p(r_0)$. $\because$ γ is a geodesic

$$\frac{\partial}{\partial v}\Big|_{v=0} r(\theta(v)) = \frac{\partial}{\partial v}\Big|_{v=0} L(\gamma_v) = -\int_0^{r_x} \langle V(r), \underbrace{\nabla_{\gamma'}\gamma'(r)}_{=0 \text{ as geod-esic}}\rangle dr + \langle V(r), \gamma'(r)\rangle_{r=0}^{r_x}$$

$$= \langle \theta'(r_x), \gamma'(r_x)\rangle$$

$$\Rightarrow \langle X, \nabla r\rangle(x) = dr(X)\big|_x = \langle X(\gamma(r_x)), \gamma'(r_x)\rangle \quad -(1)$$

$\therefore$

$$\langle X, \nabla r\rangle(x) - \langle X(p), \gamma'(0)\rangle = \int_0^{r_x} \frac{d}{dr}\langle X(\gamma(r)), \gamma'(r)\rangle dr$$

$$= \int_0^{r_x} (\nabla X)(\gamma', \gamma') dr = -\int_0^{r_x} Ric(\gamma'(r), \gamma'(r)) dr$$

$$+\frac{\lambda}{2} r_x$$

$\therefore$ we get

$$\Delta_x r(x) \leq \int_0^{r_x} \left[(n-1)(\xi')^2(r) + (1-\xi^2(r))(Ric(\gamma',\gamma')) \right] dr$$

$$-\frac{\lambda}{2} r(x) + \langle X(p), \gamma'(0) \rangle$$

Set $\xi(r) = \frac{r}{r_x}$ for $0 \leq r \leq r_0$ and $\xi(r) = 1$ for $r_0 < r \leq r_x$

$\therefore$ we get

$$\Delta_x r(x) \leq \frac{n-1}{r_0} + \frac{2}{3} r_0 S(p) - \frac{\lambda}{2} r(x) + |X(p)|.$$

$\square$

Prop For each $0 < \delta < \frac{1}{10}$ $\exists$ smooth function $\varphi = \varphi_\delta : \mathbb{R} \to [0,1]$ s.t.

$$\varphi(x) = \begin{cases} 1, & x \leq \delta \\ 0, & x \geq 2 \end{cases}, \quad -(1+\theta)\sqrt{\varphi} \leq \varphi' \leq 0, \quad |\varphi''| \leq C_0$$

and $1 - \varphi(x) + \frac{x}{2}\varphi'(x) \geq -\varepsilon$ where $\theta = \theta(\delta)$ and $\varepsilon = \varepsilon(\delta)$ are positive and they $\to 0$ as $\delta \to 0$.

$\boxed{\text{II}}$

we come back to the proof of the main theorem.

Proof:- (in the noncompact case):

Let $p \in M^n$, $r(x) = d(x,p)$. We choose $0 \leq r_0 < 1$ s.t.

$$|X(p)| \leq \frac{1}{r_0} \quad \text{and} \quad |Ric| \leq \frac{1}{r_0^2} \quad \text{on } B_{r_0}(p).$$

$\forall \; 0 < \delta < \frac{1}{10}$, $a > \frac{1}{\delta}$, define

$$\phi = \phi_\delta : M^n \to [0,1] \quad x \longmapsto \varphi\left(\frac{r(x)}{a r_0}\right) \text{ where } \varphi$$

is from the Lemma/Prop. before. Let

$$F = F_{\delta,a} = \left(\phi_{\delta,a} \cdot R\right) : M^n \longrightarrow \mathbb{R}$$

where R is the scalar curvature.

it suffices to show that at the point where $F_{\delta,a}$ achieves it minimum

Claim:-

$$F(x_0) \geq \begin{cases} -\dfrac{C_1}{a} & , \lambda \geq 0 \\[2ex] (1+\varepsilon(\delta))\dfrac{n\lambda}{2} - \dfrac{C_1}{a} & , \lambda < 0 \end{cases}$$

b/c $a \to \infty$ as $\delta \to 0$ and also $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

Also $C_1 = C_1(n, \varepsilon, \delta, r_0) > 0$.

Proof of the claim.

Case 1

Let $x_0 \in B_{\delta a r_0}(p)$. By construction $F \equiv R$ in a nbd of x_0.

Then $0 \leq \Delta_x F = \Delta_x R = -2|Ric|^2 + \lambda R$

$$= -2\langle Ric, Ric\rangle - 2\left(\frac{R^2}{n^2}\langle g, g\rangle - \frac{2R}{n}\langle g, Ric\rangle\right) - \frac{2}{n}R^2 + \lambda R$$

$$= -2\left|Ric - \frac{R}{n}g\right|^2 - \frac{2}{n}R\left(R - \frac{n\lambda}{2}\right)$$

June 03'2025

now consider $x_0 \notin B_{\delta a r_0}(p)$. If $F(x_0) \geq 0$ there is nothing to show so assume that $F(x_0) < 0$.

- $\varphi(r) = 0$ for $r \geq 2$ so $x_0 \in B_{2\delta a r_0}(p)$ and $\varphi(x_0) > 0$.

$\because$ x_0 is the point of minima $\Rightarrow$ $0 = \nabla F$

$$= (\nabla\phi)\cdot R + \phi(\nabla R) \text{ at } x_0.$$

$$-R\frac{\nabla\phi}{\varphi} \text{ at } x_0$$

we have

$$0 \leq \Delta_x F = \underbrace{\phi\,\Delta_x R}_{\leq -\frac{2}{n}R\cdot\left(R-\frac{n}{2}\lambda\right)} + 2\langle\nabla R, \nabla\phi\rangle + R\cdot\Delta_x(\varphi)$$

$$\leq -\frac{2F}{n}\left(R-\frac{n\lambda}{2}\right) - 2R\frac{|\nabla\phi|^2}{\phi} + R\cdot\Delta_x\varphi$$

$$\text{———} \quad (**)$$

we have from last time

$$\Delta_x r \leq -\frac{\lambda}{2} + C(n)\left(\underbrace{K_0 r_0}_{\leq r_0^{-1}} + r_0^{-1}\right) + \underbrace{1x1(\phi)}_{\leq r_0^{-1}}$$

by our choice of r_0

$$\leq \begin{cases} \dfrac{\tilde{C}(n)}{r_0}, & \lambda \geq 0 \\[2ex] \dfrac{\tilde{C}(n)}{r_0} - \dfrac{\lambda}{2}r, & \lambda < 0 \end{cases}$$

$\therefore$

$$\Delta_x \varphi = \frac{\varphi'}{a r_0}\Delta_x r + \frac{\varphi''}{a^2 r_0^2} \geq \begin{cases} -\frac{C_2}{a}, & d \geq 0 \\ \frac{d r \varphi'}{2 a r_0} - \frac{C_2}{a}, & d < 0. \end{cases}$$

use the fact that $|\varphi''| \leq C_0 \Rightarrow \frac{\varphi''}{a r_0^2} \geq -\tilde{C}_0$

and $\frac{\varphi'}{a \cdot r_0} \geq -\frac{\tilde{C}_2}{a}$

now, as $\phi(x) = \varphi\left(\frac{r(x)}{a r_0}\right)$ and $|\nabla r|^2 = 1$, it ~~f~~ follows

$$\langle \nabla \phi, \nabla \phi \rangle = \underbrace{\varphi'\left(\frac{r(x)}{a r_0}\right)^2}_{} \cdot \frac{1}{(a r_0)^2} \cdot |\nabla r|^2$$

$$\leq (1+\theta)^2 \cdot \phi \quad \text{as } \varphi' \leq 0.$$

By assumption, $F(x_0) = -|F|(x_0)$.

All together, for $d \geq 0$ we have

$$0 \leq -\frac{2F}{n}\left(R - \frac{n d}{2}\right) - 2R\frac{|\nabla \phi|^2}{\phi} + R\,\Delta_x \phi$$

$$\leq -\frac{2F}{n}\Big(\underbrace{\phi R}_{F} - \frac{n\lambda\phi}{2}\Big) - \frac{2}{n\phi}\cdot n\cdot R\cdot|\nabla\phi|^2$$

$$- \frac{2\phi R}{n\phi}\cdot\frac{n}{2}\left(-\Delta_x\phi\right) \quad \text{—— } (I)$$

$$\leq \frac{2|F|}{n\phi}\left(F - \frac{n\lambda\phi}{2} + \frac{n(1+\theta)^2}{a^2 r_0^2} + \frac{nC_2}{2a}\right)$$

$$\leq \frac{2|F|}{n\phi}\left(F + \frac{C_3(n,\delta,r_0)}{a}\right)$$

$$\Rightarrow \quad F(x_0) \geq -\frac{C_3}{a}$$ thus proving the claim for $\lambda \geq 0$.

when $\lambda < 0$,

$$0 \leq \Delta_x F \leq (I) \leq \frac{2|F|}{n\phi}\Big(F - \frac{n\lambda\phi}{2} + \frac{n(1+\theta)^2}{a^2 r_0^2} + \frac{nC_2}{2a} + \frac{n\lambda\phi' r}{4ar_0}\Big)$$

$$\leq \frac{2|F|}{n\phi}\left(F+\frac{C_3}{a}-\frac{n\lambda}{2}\left(\varphi-\frac{\varphi' r}{2ar_0}\right)\right)$$

$$\leq \frac{2|F|}{n\phi}\left(F+\frac{C_3}{a}-\frac{n\lambda}{2}+\frac{n\lambda}{2}\left(1-\varphi+\frac{\varphi' r}{2ar_0}\right)\right)$$

By construction of φ we know that

$$1-\varphi\left(\frac{r}{ar_0}\right)+\frac{r}{2ar_0}\varphi'\left(\frac{r}{ar_0}\right)\geq -\varepsilon(\delta)$$

$\Rightarrow$

$$\leq \frac{2|F|}{n\phi}\left(F+\frac{C_3}{a}-\frac{n\lambda}{2}\left(1+\varepsilon(\delta)\right)\right)$$

$\therefore$ at x_0 we get

$$F(x_0)\geq \frac{n\lambda}{2}\left(1+\varepsilon(\delta)\right)-\frac{C_3}{a}$$ as the claim.

and so we get the estimate on R.

The equality case:-

Gaussian shrinker:- $(\mathbb{R}^n, g_{Eucl}, f_{Gauss}, \lambda)$

$$f_{Gauss}(x) := \frac{\lambda}{4}|x|^2.$$

assume R achieves the equality i.e. the equality at some point $p \in M$, i.e, $R(p) = 0$ $\lambda \geq 0$

$$R(p) \leq \frac{n\lambda}{2}, \ \lambda < 0.$$

$\Longrightarrow$ by the strong maximum principle R is constant and $R(x) = 0$ or $R(x) = \frac{n\lambda}{2}$ $\forall x$.

$$\therefore 0 \leq \Delta_X R = -2\left|Ric - \frac{R}{n}g\right|^2 - \frac{2}{n}\underbrace{R\left(R - \frac{n\lambda}{2}\right)}_{0 \text{ in both cases}}$$

$\Longrightarrow$ $Ric = \frac{R}{n}g$ $\Longrightarrow$ g is Einstein.

now assume $\lambda > 0$ and $X = \nabla f$, $f \in C^\infty(M)$.
WLOG, let $\lambda = 1$. We just showed that (M^n, g) is Einstein and $R = 0$ at some point $\Rightarrow Ric \equiv 0$.

from $$0 \le \underbrace{R}_{=0} + \underbrace{|\nabla f|^2}_{\ge 0} = \lambda f$$

and $$\nabla^2 f + Ric = \frac{\lambda}{2} g = \frac{1}{2} g > 0.$$

cf. Thm 4.3

$\Rightarrow$ f attains its minima which is unique ~~and~~
$\therefore$ $\nabla f(x_0) = 0$ at the minimum point x_0
$\Rightarrow$ $f(x_0) = 0$. cf. Prop. 2.9

$\Rightarrow$ $f > 0$ on $M \setminus \{x_0\}$.

if we define $\rho = 2\sqrt{f}$ on $M \setminus \{x_0\}$.
then
$$\nabla^2(\rho^2) = 2g \Rightarrow |\nabla \rho|^2 = \left|\frac{2}{2\sqrt{f}} \nabla f\right|^2$$
$$= \frac{1}{f} |\nabla f|^2 = 1.$$

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